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HW1

A ME 549
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1. Consider a simplified model of the Newton's cradle consisting of a pair of pendulums with mass m_L, m_R.

   (a) Determine the state of the system

   Solution. For simplicity, assume that m_L, m_R are point masses, so that upon colliding, both angles $x_1, x_3 = 0$. The state of the system can be determined by the two angles x_1, x_3 and their respective rates of change. Therefore, the state vector is

$$\mathbf{z} = \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_3 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

   (b) Define each element of a hybrid system model (C, f, D, g) assuming the following:

   - Viscous friction for circular motion.
   - Conservation of momentum at impacts.
   - Dissipation of energy at impacts.

   Solution. To define C, first define the following two sets:

$$\begin{aligned} C_\theta &= \{\mathbf{z} \in \mathbb{R}^4 \text{ s.t. } -\pi/2 \leq z_1, z_3 \leq \pi/2 \text{ and } z_1 < z_3\} \\ D &= \{\mathbf{z} \in C_\theta \text{ s.t. } |z_1 - z_3| = d \text{ and } (z_2 - z_4) > 0\} \end{aligned}$$

   For $z_1 < z_3$ in the definition of C_θ, this assumes the ball diameter is greater than zero. In words, the set C_θ represents a "ceiling" condition, so that neither pendulum L or R can cross over and flip its position. The set D represents the set of all possible points where balls L and R collide, i.e. the jump set. In words, considering d the diameter of the balls, when the balls are in contact and have a positive relative velocity (they are moving towards each other) a collision occurs. The flow set is then the set C_θ discarding all points in D.

$$\begin{aligned} C &= C_\theta \setminus D \\ &= C_\theta \cap D^C. \end{aligned}$$

   The flow mapping is given by the standard equations for a damped pendulum (assuming that $l_L = l_R = 1$ for simplicity, $g = 9.81$, the standard gravity constant).

$$f(\mathbf{z}) = \begin{pmatrix} z_2 \\ -g\sin(z_1) - c_L z_2 \\ z_4 \\ -g\sin(z_3) - c_R z_4 \end{pmatrix}.$$

The jump map will take into account the conservation of momentum and dissipation at impacts. Assume the collisions are instantaneous, but "sticky". That is, assuming the case of the left ball colliding with the right (the right ball will have zero velocity at this point). The following shows conservation of momentum, and dissipation of energy at impacts (thinking of $K = 1/2mv^2$)

$$\begin{aligned} m_L z_2 &= (m_L + m_R) z_4 \\ \implies z_4 &\mapsto (\frac{m_L}{m_L + m_R}) z_2 \end{aligned}$$

Extending the logic to when both balls are potentially in motion towards one another:

$$g(\mathbf{z}) = \begin{pmatrix} z_1 \\ (\frac{\lambda m_R}{m_L + m_R}) z_4 \\ z_3 \\ (\frac{\lambda m_L}{m_L + m_R}) z_2 \end{pmatrix}.$$

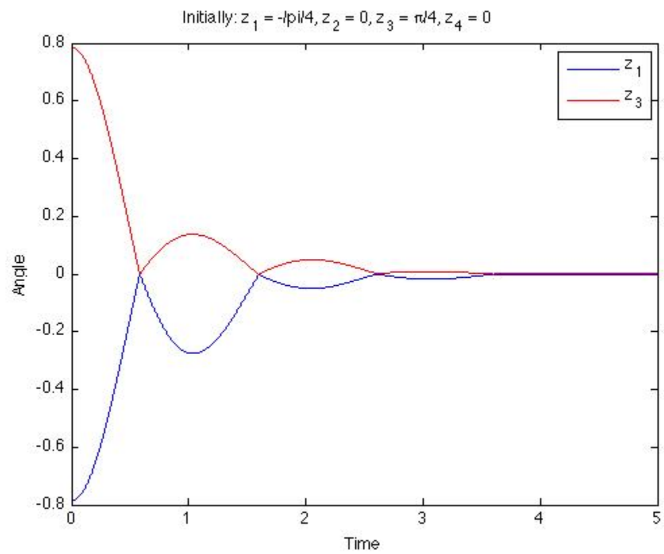
Note the introduction of λ, the coefficient of restitution. This factor (< 1) will provide additional energy dissipation at impact, though momentum will no longer be conserved.

(c) Perform the following simulations of the resulting hybrid system for parameters $m_L = 1$, $m_R = 2$, unitary viscous friction, and a restitution law with parameters such that there is dissipation at impacts:

Solution. Note: For all simulations, $c_L = c_R = 1$ and $\lambda = 0.9$. For the numerics, assume a small d, and a small enough time step to ensure collisions aren't "flowed" over.

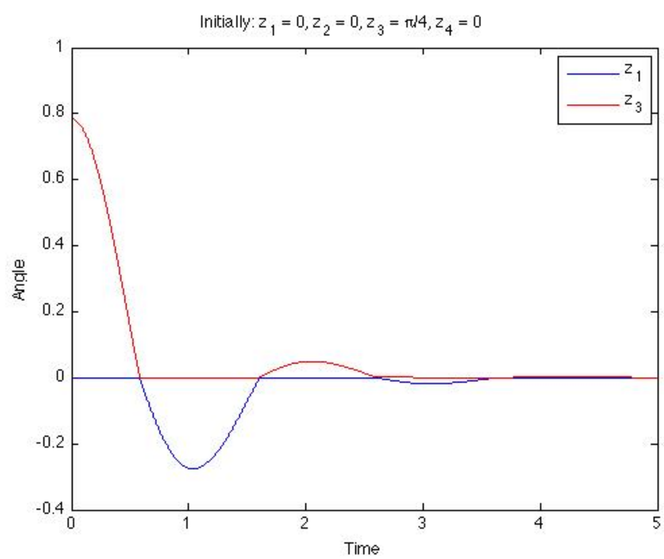
i. Plot trajectories as a function of t for pendulums starting at $x_1 = -\pi/4$ and $x_3 = \pi/4$ with zero velocity.

Solution. The figure is shown below. Observe the neat beat phenomena ☺.



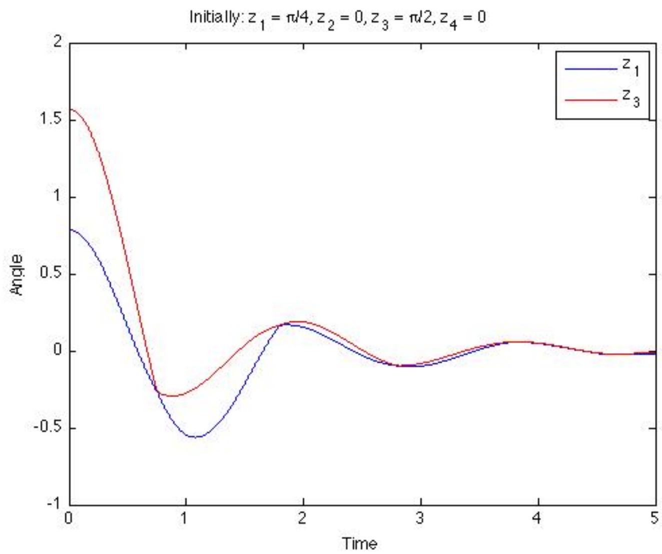
ii. Plot trajectories as a function of t for pendulums starting at $x_1 = 0$ and $x_3 = \pi/4$ with zero velocity.

Solution.



iii. Plot trajectories as a function of t for pendulums starting at $x_1 = \pi/4$ and $x_3 = \pi/2$ with zero velocity.

Solution.



2. Consider the implementation of a static controller

$$\kappa : \mathbb{R}^n \to \mathbb{R}^m$$

for the continuous-time plant

$$\dot{\xi} = \tilde{f}(\xi, u)$$

in a *digital device*, e.g. computer, micro controller, digital signal processor, etc. This is depicted in Figure 2 on Homework 1, where the controller is interfaced with sample-and-hold devices. The sample-and-hold devices. The sample-and-hold device that samples the state ξ of the plant is referred to as a *sampling device* (or analog-to-digital (A/D) converter), while the sample-and-hold device that stores the output of the controller in between computations is referred to as a *hold device* (or digital-to-analog (D/A) converter) which is assumed to be of zero-order type, that is, a zero-order hold (ZOH).

(a) Define each element of a hybrid system (C, f, D, g) and its state assuming the following

- The computation of the feedback law takes no time, i.e. is instantaneous.
- The positive constants T_s and T_c are not necessarily equal.

Solution. Assume WLOG $T_s < T_c$. I say WLOG, since the argument follows the same either way. In order to develop $\mathcal{H}$, I will set the state vector and define each component.

$$\mathbf{x} \equiv \begin{pmatrix} \xi \\ \hat{\xi} \\ u \\ \tau_s \\ \tau_c \end{pmatrix}$$

In the above definition, ξ is the continuous state of the nonlinear system, $\hat{\xi}$ is the sampled value of the state ξ, u is the input, τ_s is the timer for the sample device, and τ_c is the timer for the hold device. The entire state space for the system would then be $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. However, since I intend to reset each timer once the sample or hold period has been met, I make the following restriction (which gives the flow set C)

$$C = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times [0, T_s) \times [0, T_c).$$

Now, jumps will occur whenever either timer hits its period limit, so

$$D = D_s \cup D_c$$

where

$$\begin{aligned} D_s &= \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \{T_s\} \times [0, T_c) \\ D_c &= \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times [0, T_s) \times \{T_c\}. \end{aligned}$$

For the flow map,

$$f(\mathbf{x}) = \begin{pmatrix} \tilde{f}(\xi, u) \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

This is saying that, during flow periods, the timers scale linearly with time (well, duh), ξ evolves continuously, and the sampled and input values don't change.

For the jumps, there are two different times (T_s, T_c) to consider. When $\tau_s = T_s$, the timer τ_s resets to zero and the sampled value updates to the present value of the system state ξ, which results in the following jump map

$$g_s(\mathbf{x}) = \begin{pmatrix} \xi \\ \xi \\ u \\ 0 \\ \tau_c \end{pmatrix}.$$

When, $\tau_c = T_c$, the timer τ_c resets to zero and the input value updates according to the feedback law. This results in the jump map

$$g_c(\mathbf{x}) = \begin{pmatrix} \xi \\ \hat{\xi} \\ \kappa(\hat{\xi}) \\ \tau_s \\ 0 \end{pmatrix}.$$

The entire jump map formulation is

$$g(\mathbf{x}) = \begin{pmatrix} g_s(\mathbf{x}) \text{ if } \mathbf{x} \in D_s \\ g_c(\mathbf{x}) \text{ if } \mathbf{x} \in D_c \end{pmatrix}.$$

(b) Explain how the model would change if the computation of the control law takes $\delta > 0$ units of time.

Solution. I believe that the addition of a finite (non-zero) amount of time for the control law would require the addition of an additional counter τ_δ that would reset each time it hit δ. At this time, the value of the input to the hold device would change to the updated value of $\kappa(\hat{\xi})$. It would probably be best to add an additional state to $\mathbf{x}$ that monitored the value being held. Additionally, there will be an additional jump condition each time the τ_δ timer reset. The argument would follow much the same as the one I made in part a).

3. Select a hybrid system that you are familiar with and that is not in the course material, explain why it is hybrid, describe its behavior and model it as a hybrid system (C, f, D, g). Pick an appropriate set of parameters for your system and provide trajectories for **two different** sets of initial conditions.

Solution. For simplicity's sake, consider a simple harmonic oscillator with a natural frequency that switches each time the velocity changes sign. The system is clearly hybrid, since there is a discontinuity in the augmented state (position,velocity, and natural frequency) at each velocity crossing. The behavior will still be oscillatory, but the period of motion will change due to the shift in natural frequency. The flow map will be

$$f(\mathbf{z}) = \begin{pmatrix} z_2 \\ -z_3^2 z_1 \\ 0 \end{pmatrix}$$

where

$$\mathbf{z} = \begin{pmatrix} x \\ \dot{x} \\ \omega_0 \end{pmatrix}.$$

This mapping is valid whenever the velocity is not changing signs (i.e. not equal to zero). So, $C = \mathbb{R} \times (\mathbb{R} \cap \{0\})^C \times \mathbb{R}_{\geq 0}$ The jump map could be anything, but let's just say that the natural frequency reduces by a factor of 0.6 each time the velocity changes sign

$$g(\mathbf{z}) = \begin{pmatrix} z_1 \\ z_2 \\ 0.6 z_3 \end{pmatrix}$$

This mapping is valid for any $\mathbf{z} \in D$, where $D = \mathbb{R} \times \{0\} \times \mathbb{R}_{\geq 0}$. To ensure that the system starts to flow again after the jump map, I put in a counter variable that resets each time a jump occurs. This enabled me to use the hybrid simulation lite routine to run the simulations.

Plots for two separate initial conditions are shown in the figure below, where

$$\text{IC1} = \begin{pmatrix} 1.0 \\ 0.0 \\ 10.0 \end{pmatrix}, \text{IC2} = \begin{pmatrix} 0.0 \\ 2.0 \\ 14.0 \end{pmatrix}$$